MAGIC SQUARES.

[CONCLUSION.]

EVEN MAGIC SQUARES.

THE writer believes that the method of constructing even magic squares by a predetermined geometrical arrangement of numbers is new and original. It will be noted that the foregoing dia-

1	195	3	193	5	191	190	7	188	10	186	12	184	14
182	16	180	18	178	20	176	175	23	173	25	171	27	15
168	167	31	165	33	163	35	36	160	38	158	40	30	155
43	153	152	46	150	48	148	147	51	145	53	45	142	56
140	58	138	137	61	135	63	64	132	66	60	129	69	127
71	125	73	123	122	76	120	119	79	75	116	82	114	84
112	86	110	88	108	107	91	92	90	103	95	101	97	99
98	100	96	102	89	93	105	106	104	94	109	87	111	85
113	83	115	74	80	118	78	77	121	117	81	124	72	126
70	128	59	67	131	65	133	134	62	136	130	68	139	57
141	44	54	144	52	146	50	49	149	47	151	143	55	154
29	41	157	39	159	37	161	162	34	164	32	166	156	42
28	170	26	172	24	174	21	22	177	19	179	17	181	169
183	13	185	11	187	6	8	189	9	192	4	194	2	196

Fig. 66.

grams illustrate in a graphic manner the interesting results attained by the harmonious association of figures, and they also clearly demonstrate the almost infinite variety of possible combinations.

THE CONSTRUCTION OF EVEN MAGIC SQUARES BY DE LA HIRE'S METHOD.

A perfect magic square of 4×4 may be constructed as follows:

1. Fill the corner diagonal columns of a 4×4 square with the numbers 1 to 4 in arithmetical sequence, starting from the upper and lower left hand corners (Fig. 67).
2. Fill the remaining empty cells with the missing numbers of the series 1 to 4 so that the sum of every perpendicular and horizontal column equals 10 (Fig. 68).

1			4
	2	3	
	2	3	
1			4

Fig. 67.

1	3	2	4
4	2	3	1
4	2	3	1
1	3	2	4

Fig. 68.

1	4	4	1
3	2	2	3
2	3	3	2
4	1	1	4

Fig. 69.

3. Construct another 4×4 square, having all numbers in the same positions relatively to each other as in the last square, but reversing the direction of all horizontal and perpendicular columns (Fig. 69).
4. Form the key square Fig. 70 from Fig. 69 by substituting key numbers for prime numbers, and then add the numbers in this key square to similarly located numbers in the primary square Fig. 68. The result will be the perfect square of 4×4 shown in Fig. 72.

By making the key square Fig. 71 from the primary square Fig. 68 and adding the numbers therein to similarly located numbers in the primary square Fig. 69, the same magic square of 4×4 will be produced, but with all horizontal and perpendicular columns reversed in direction as shown in Fig. 73.

The magic square of 6×6 shown in Figure 46 and also a large number of variations of same may be readily constructed by

the De la Hire method, and the easiest way to explain the process will be to analyze the above mentioned square into the necessary primary and key squares, using the prime numbers 1 to 6 with their respective key numbers as follows:

Prime numbers 1, 2, 3, 4, 5, 6.
Key numbers 0, 6, 12, 18, 24, 30.

The cells of two 6×6 squares may be respectively filled with prime and key numbers by analyzing the contents of each cell in Fig. 46. Commencing at the left hand cell in the upper row, we note that this cell contains 1. In order to produce this number by the addition of a prime number to a key number it is evident that

PRIME NUMBERS	KEY NUMRERS
1	0
2	4
3	8
4	12

0	12	12	0
8	4	4	8
4	8	8	4
12	0	0	12

Fig. 70.

0	8	4	12
12	4	8	0
12	4	8	0
0	8	4	12

Fig. 71.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Fig. 72.

1	12	8	13
15	6	10	3
14	7	11	2
4	9	5	16

Fig. 73.

0 and 1 must be selected and written into their respective cells. The second number in the top row of Fig. 46 being 35, the key number 30 must be written in the second cell of the key square and the prime number 5 in the second cell of the prime square, and so on throughout all the cells, the finished squares being shown in Figs. 74 and 75.

Another prime square may now be derived from the key square Fig. 74 by writing into the various cells of the former the prime numbers that correspond to the key numbers of the latter. This second primary square is shown in Fig. 76. It will be seen that the numbers in Fig. 76 occupy the same relative positions to each other

as the numbers of the first primary square (Fig. 75), but the direction of all columns is changed from horizontal to perpendicular, and vice versa.

To distinguish and identify the two primary squares which are used in these operations, the first one (in this case Fig. 75) will in future be termed the A primary square, and the second one (in this case Fig. 76) the B primary square.

1	35	34	3	32	6
30	8	28	27	11	7
24	23	15	16	14	19
13	17	21	22	20	18
12	26	9	10	29	25
31	2	4	33	5	36

Fig. 46 (Dup.)

0	30	30	0	30	0
24	6	24	24	6	6
18	18	12	12	12	18
12	12	18	18	18	12
6	24	6	6	24	24
30	0	0	30	0	30

Fig. 74.

It is evident that the magic square of 6×6 shown in Fig. 46 may now be reconstructed by adding the cell numbers in Fig. 74

1	5	4	3	2	6
6	2	4	3	5	1
6	5	3	4	2	1
1	5	3	4	2	6
6	2	3	4	5	1
1	2	4	3	5	6

Fig. 75.

1	6	6	1	6	1
5	2	5	5	2	2
4	4	3	3	3	4
3	3	4	4	4	3
2	5	2	2	5	5
6	1	1	6	1	6

Fig. 76.

to the similarly placed cell numbers in Fig. 75. Having thus inversely traced the development of the magic square from its A and B primary and key squares, it will be useful to note some of the general characteristics of even primary squares, and also to study the rules which govern their construction, as these rules will be found instructive in assisting the student to work out an almost endless variety of even magic squares of all dimensions.

1. Referring to the 6×6 A primary square shown in Fig. 75, it will be noted that the two corner diagonal columns contain the numbers 1 to 6 in arithmetical order, starting respectively from the upper and lower left hand corner cells, and that the diagonal columns of the B primary square in Fig. 76 also contain the same numbers in arithmetical order but starting from the two upper corner cells. The numbers in the two corner diagonal columns are subject to many arrangements which differ from the above but it will be unnecessary to consider them in the present article.
2. The numbers in the A primary square Fig. 75 have the same relative arrangement as those in the B primary square Fig. 76, but the horizontal columns in one square form the perpendicular columns in the other and vice versa. This is a general but not a universal relationship between A and B primary squares.
3. The sum of the series 1 to 6 is 21 and the sum of every column in both A and B 6×6 primary squares must also be 21.
4. The sum of every column in a 6×6 key square must be 90, and under these conditions it follows that the sum of every column of a 6×6 magic square which is formed by the combination of a primary square with a key square must be 111 $(21 + 90 = 111)$.
5. With the necessary changes in numbers the above rules hold good for all sizes of A and B primary squares and key squares.

We may now proceed to show how a variety of 6×6 magic squares can be produced by different combinations of numbers in primary and key squares. The six horizontal columns in Fig. 75 show some of the combinations of numbers from 1 to 6 that can be used in 6×6 A primary squares, and the positions of these columns or rows of figures relatively to each other may be changed so as to produce a vast variety of squares which will naturally lead to the development of a corresponding number of 6×6 magic squares.

In order to illustrate this in a systematic manner the different rows of figures in Fig. 75 may be rearranged and identified by letters as given in Fig. 77.

a	1	2	4	3	5	6
b	1	5	4	3	2	6
c	1	5	3	4	2	6
d	6	5	3	4	2	1
e	6	2	3	4	5	1
f	6	2	4	3	5	1

Fig. 77.

Fig. 78 shows the sequence of numbers in the diagonal columns of these 6×6 A primary squares, and as this arrangement cannot

1st line	1					6	*a, b,* or *c.*
2nd "		2			5		*a, e,* or *f.*
3rd "			3	4			*c, d,* or *e.*
4th "			3	4			*c, d,* or *e.*
5th "		2			5		*a, e,* or *f.*
6th "	1					6	*a, b,* or *c.*

Fig. 78.

be changed in this series, the various horizontal columns or rows in Fig. 77 must be selected accordingly. The small letters at the right

No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.
a	*a*	*b*	*b*	*c*	*c*
f	*e*	*f*	*e*	*a*	*f*
c	*d*	*c*	*d*	*d*	*e*
d	*c*	*d*	*c*	*e*	*d*
e	*f*	*e*	*f*	*f*	*a*
b	*b*	*a*	*a*	*b*	*b*

Fig. 79.

of Fig. 78 indicate the different horizontal columns that may be used for the respective lines in the square; thus either *a, b,* or *c* column

in Fig. 77 may be used for the first and sixth lines, *a, e,* or *f* for the second and fifth, and *c, d,* or *e* for the third and fourth lines, but neither *b, c,* or *d* can be used in the second or fifth lines, and so forth.

Six different combinations of columns are given in Fig, 79. from which twelve different 6×6 magic squares may be constructed. Taking column No. 1 as an example, Fig. 80 shows an

a	1	2	4	3	5	6
f	6	2	4	3	5	1
c	1	5	3	4	2	6
d	6	5	3	4	2	1
e	6	2	3	4	5	1
b	1	5	4	3	2	6

Fig. 80.

1	6	1	6	6	1
2	2	5	5	2	5
4	4	3	3	3	4
3	3	4	4	4	3
5	5	2	2	5	2
6	1	6	1	1	6

Fig. 81.

A primary square made from the combination *a, f, c, d, e, b,* and Fig. 81 is the B primary square formed by reversing the direction of the horizontal and perpendicular columns of Fig. 80. The key square Fig. 82 is then made from Fig. 81 and the 6×6 magic

0	30	0	30	30	0
6	6	24	24	6	24
18	18	12	12	12	18
12	12	18	18	18	12
24	24	6	6	24	6
30	0	30	0	0	30

Fig. 82.

0	30	30	0	30	0
24	6	24	24	6	6
18	12	12	12	18	18
12	18	18	18	12	12
6	24	6	6	24	24
30	0	0	30	0	30

Fig. 83.

square in Fig. 84 is the result of adding the cell numbers of Fig. 82 to the corresponding cell numbers in Fig. 80.

The above operation may be varied by reversing the horizontal columns of the key square Fig. 82 right and left as shown in Fig. 83 and then forming the magic square given in Fig. 85. In this way two different magic squares may be derived from each combination.

It will be noted that all the 6×6 magic squares that are constructed by these rules are similar in their general characteristics to the 6×6 squares which are built up by the diagrammatic system.

Perfect 8×8 magic squares may be constructed in great variety by the method now under consideration, and the different com-

1	32	4	33	35	6
12	8	28	27	11	25
19	23	15	16	14	24
18	17	21	22	20	13
30	26	9	10	29	7
31	5	34	3	2	36

Fig. 84.

1	32	34	3	35	6
30	8	28	27	11	7
19	17	15	16	20	24
18	23	21	22	14	13
12	26	9	10	29	25
31	5	4	33	2	36

Fig. 85.

binations of numbers from 1 to 8 given in Fig. 86 will be found useful by laying out a large number of A primary squares.

1	7	6	4	5	3	2	8	*a*
1	2	6	4	5	3	7	8	*b*
1	2	6	5	4	3	7	8	*c*
1	7	3	4	5	6	2	8	*d*
1	7	3	5	4	6	2	8	*e*
8	2	3	5	4	6	7	1	*aa*
8	7	3	5	4	6	2	1	*bb*
8	7	3	4	5	6	2	1	*cc*
8	2	6	5	4	3	7	1	*dd*
8	2	6	4	5	3	7	1	*ee*

Fig. 86.

Fig. 87 shows the fixed numbers in the diagonal columns of these 8×8 A primary squares, and also designates by letters the specific rows of figures which may be used for the different horizontal columns. Thus the row marked *a* in Fig. 86 may be used for the first, fourth, fifth, and eighth horizontal columns but cannot

be employed for the second, third, sixth or seventh columns, and so forth.

Fig. 88 suggests half a dozen combinations which will form as many primary squares, and it is evident that the number of possible variations is very large. It will suffice to develop the first and third of the series in Fig. 88 as examples.

1st line	1							8	*a, b, c, d,* or *e.*
2nd "		2					7		*b, c, aa, dd,* or *ee.*
3rd "			3			6			*d, e, aa,* or *cc.*
4th "				4	5				*a, b, d, cc,* or *ee.*
5th "				4	5				*a, b, d, cc,* or *ee.*
6th "			3			6			*d, e, aa,* or *cc.*
7th "		2					7		*b, c, aa, dd,* or *ee.*
8th "	1							8	*a, b, c, d,* or *e.*

Fig. 87.

Fig. 89 is the A primary square developed from column No. 1 in Fig. 88, and Fig. 90 is the B primary square made by reversing

No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.
a	*b*	*c*	*d*	*e*	*a*
aa	*b*	*c*	*dd*	*ee*	*b*
aa	*d*	*cc*	*e*	*e*	*e*
a	*b*	*cc*	*d*	*ee*	*d*
a	*b*	*cc*	*d*	*ee*	*d*
aa	*d*	*cc*	*e*	*e*	*e*
aa	*b*	*c*	*dd*	*ee*	*b*
a	*b*	*c*	*d*	*e*	*a*

Fig. 88.

the direction of all horizontal and perpendicular columns of Fig. 89. Substituting key numbers for the prime numbers in Fig. 90, and adding these key numbers to the prime numbers in Fig. 89 gives the perfect magic square of 8 × 8 shown in Fig. 91. The latter will be found identical with the square which may be written out directly from diagrams in Fig. 52.

Fig. 92 shows an A primary square produced from column No. 3 in Fig. 88. The B primary square Fig. 93 being made in the regular way by reversing the direction of the columns in Fig. 92.

Prime numbers 1, 2, 3, 4, 5, 6, 7, 8.
Key numbers 0, 8, 16, 24, 32, 40, 48, 56.

1	7	6	4	5	3	2	8	a
8	2	3	5	4	6	7	1	aa
8	2	3	5	4	6	7	1	aa
1	7	6	4	5	3	2	8	a
1	7	6	4	5	3	2	8	a
8	2	3	5	4	6	7	1	aa
8	2	3	5	4	6	7	1	aa
1	7	6	4	5	3	2	8	a

Fig. 89.

1	8	8	1	1	8	8	1
7	2	2	7	7	2	2	7
6	3	3	6	6	3	3	6
4	5	5	4	4	5	5	4
5	4	4	5	5	4	4	5
3	6	6	3	3	6	6	3
2	7	7	2	2	7	7	2
8	1	1	8	8	1	1	8

Fig. 90.

The perfect magic square of 8×8 in Fig. 94 is developed from these two primary squares as in the last example, and it will be

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

Totals = 260.

Fig. 91.

found similar to the square which may be formed directly from diagram No. 2 in Fig. 54.

Fig. 95 shows another 8×8 magic square which is constructed by combining the A primary square in Fig. 89 with the B primary square in Fig. 93 after changing the latter to a key square in the

manner before described. This magic square may also be directly constructed from diagram No. 4 in Fig. 54.

It is evident that an almost unlimited number of different 8×8 magic squares may be made by the foregoing methods, and

1	2	6	5	4	3	7	8	c
1	2	6	5	4	3	7	8	c
8	7	3	4	5	6	2	1	cc
8	7	3	4	5	6	2	1	cc
8	7	3	4	5	6	2	1	cc
8	7	3	4	5	6	2	1	cc
1	2	6	5	4	3	7	8	c
1	2	6	5	4	3	7	8	c

Fig. 92.

1	1	8	8	8	8	1	1
2	2	7	7	7	7	2	2
6	6	3	3	3	3	6	6
5	5	4	4	4	4	5	5
4	4	5	5	5	5	4	4
3	3	6	6	6	6	3	3
7	7	2	2	2	2	7	7
8	8	1	1	1	1	8	8

Fig. 93.

their application to the formation of other and larger squares is so obvious that it will be unnecessary to present any further examples.

1	2	62	61	60	59	7	8
9	10	54	53	52	51	15	16
48	47	19	20	21	22	42	41
40	39	27	28	29	30	34	33
32	31	35	36	37	38	26	25
24	23	43	44	45	46	18	17
49	50	14	13	12	11	55	56
57	58	6	5	4	3	63	64

Fig. 94.

1	7	62	60	61	59	2	8
16	10	51	53	52	54	15	9
48	42	19	21	20	22	47	41
33	39	30	28	29	27	24	40
25	31	38	36	37	35	26	32
24	18	43	45	44	46	23	17
56	50	11	13	12	14	55	49
57	63	6	4	5	3	58	64

Fig. 95.

COMPOUND MAGIC SQUARES.

The writer believes that these highly ingenious combinations were first devised by Prof. Hermann Schubert.

They may be described as a series of small magic squares arranged quadratically in magic square order.

The 9×9 square shown in Fig. 96 is the smallest of this class that can be constructed and it consists of nine 3×3 sub-squares arranged in the same order as the numerals 1 to 9 inclusive in the 3×3 square shown in Fig. 1. The first sub-square occupies the

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

Totals = 369.

Fig. 96.

middle section of the first horizontal row of sub-squares, and it contains the numbers 1 to 9 inclusive arranged in regular magic

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

Totals = 369.

Fig. 97.

square order being a duplicate of Fig. 1. The second sub-square is located in the right hand lower corner of the third horizontal row of sub-squares and it contains the numbers 10 to 18 inclusive arranged in magic square order, and so on to the last sub-square

which occupies the middle section of the third horizontal row of sub-squares, and which contains the numbers 73 to 81 inclusive.

This peculiar arrangement of the numbers 1 to 81 inclusive forms a magic square in which the characteristics of the ordinary 9×9 square are multiplied to a remarkable extent, for whereas in the latter square (Fig. 97) there are only twenty columns which sum up to 369, in the compound square of 9×9 there are an immense number of combination columns which yield this amount. This is evident from the fact that there are eight columns in the first sub-square which yield the number 15; also eight columns in

113	127	126	116	1	15	14	4	81	95	94	84
124	118	119	121	12	6	7	9	92	86	87	89
120	122	123	117	8	10	11	5	88	90	91	85
125	115	114	128	13	3	2	16	93	83	82	96
33	47	46	36	65	79	78	68	97	111	110	100
44	38	39	41	76	70	71	73	108	102	103	105
40	42	43	37	72	74	75	69	104	106	107	101
45	35	34	48	77	67	66	80	109	99	98	112
49	63	62	52	129	143	142	132	17	31	30	20
60	54	55	57	140	134	135	137	28	22	23	25
56	58	59	53	136	138	139	133	24	26	27	21
61	51	50	64	141	131	130	144	29	19	18	32

Totals = 870.

Fig. 98.

the middle sub-square which yield the number 123—and eight columns in the last sub-square which sum up to the number 231—and $15 + 123 + 231 = 369$.

The next compound square is that of 12×12 which may be built with sixteen sub-squares of 3×3 or with nine sub-squares of 4×4 the latter arrangement being shown in Fig. 98.

The next larger square of this class is that of 16×16 which can only be built with sixteen sub-squares of 4×4. Next comes the 18×18 compound square which may be constructed with

thirty-six sub-squares of 3×3 or with nine sub-squares of 6×6, and so on indefinitely with larger and larger compound squares.

CONCENTRIC MAGIC SQUARES.

Beginning with a small central magic square it is possible to arrange one or more panels of numbers concentrically around it so that after the addition of each panel, the enlarged square will still retain magic qualifications.

Either a 3×3 or a 4×4 magic square may be used as a

23	1	2	20	19
22	16	9	14	4
5	11	13	15	21
8	12	17	10	18
7	25	24	6	3

Fig. 99.

23	1	2	20	19
22	12	11	16	4
5	17	13	9	21
8	10	15	14	18
7	25	24	6	3

Fig. 102.

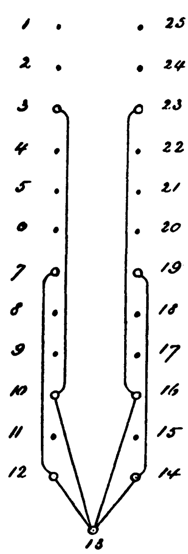

Fig. 100.

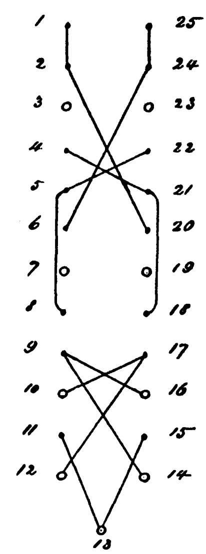

Fig. 101.

Totals of 3×3 squares $= 39$.
Totals of 5×5 squares $= 65$.

nucleus, and the square will obviously remain either odd or even, according to its beginning, irrespective of the number of panels which may be successively added to it. The center square will naturally be perfect, but after one or more panels have been added the enlarged square will no longer retain perfect characteristics, because the peculiar features of its construction will not permit the sum of every pair of geometrically opposite numbers to equal the

sum of the first and last numbers of the series used. The sum of every horizontal and perpendicular column and of the two corner diagonal columns will, however, be the same amount.

The smallest concentric square that can be constructed is that of 5×5, an example of which is illustrated in Fig. 99.

The center square of 3×3 begins with 9 and continues, with increments of 1, up to 17, the center number being 13 in accordance with the general rule for a 5×5 square made with the series of

19	2	20	1	23
4	16	9	14	22
18	11	13	15	8
21	12	17	10	5
3	24	6	25	7

Fig. 103.

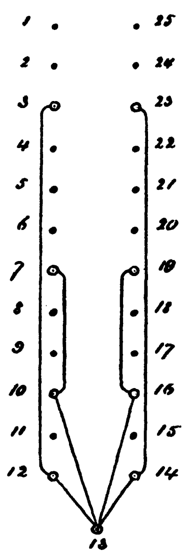

Fig. 104.

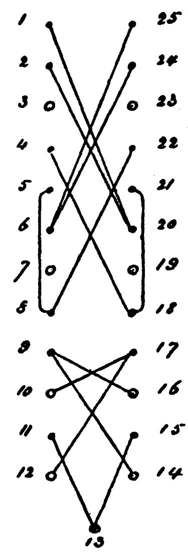

Fig. 105.

Totals of 3×3 square $= 39$.
Totals of 5×5 square $= 65$.

numbers 1 to 25. The development of the two corner diagonal columns is given in diagram Fig. 100, the numbers for these columns being indicated by small circles. The proper sequence of the other twelve numbers in the panels is shown in Fig. 101. The relative positions of the nine numbers in the central 3×3 square cannot be changed, but the entire square may be inverted or turned one quarter, one half, or three quarters around, so as to vary the position of the numbers in it relatively to the surrounding panel

numbers. Fig. 102 shows a 5×5 concentric square in which the panel numbers occupy the same cells as in Fig. 99, but the central 3×3 square is turned around one quarter of a revolution to the right.

Several variations may also be made in the location of the panel numbers, an example being given in Figs. 103, 104, and 105. Many

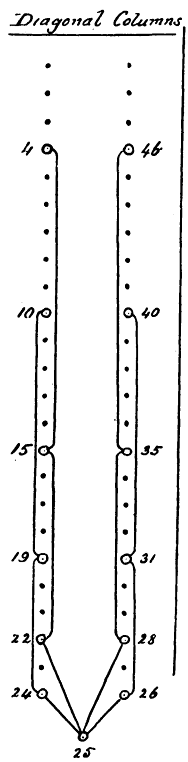

Fig. 106.

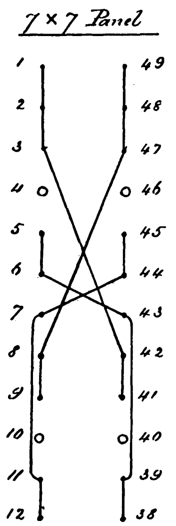

Fig 107.

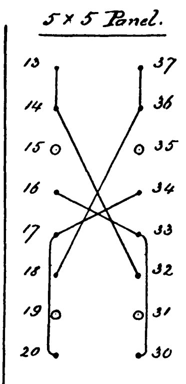

Fig. 108.

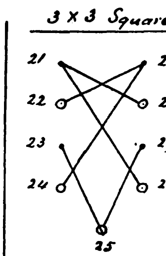

Fig. 109.

46	1	2	3	42	41	40
45	35	13	14	32	31	5
44	34	28	21	26	16	6
7	17	23	25	27	33	43
11	20	24	29	22	30	39
12	19	37	36	18	15	38
10	49	48	47	8	9	4

Fig. 110.

Totals of 3×3 square $= 75$
Totals of 5×5 square $= 125$
Totals of 7×7 square $= 175$

other changes in the relative positions of the panel numbers are selfevident.

One of many variations of the 7×7 concentric magic square

is shown in Fig. 110. The 3×3 central square in this example is started with 21 and finished with 29 in order to comply with the general rule that 25 must occupy the center cell in a 7×7 square

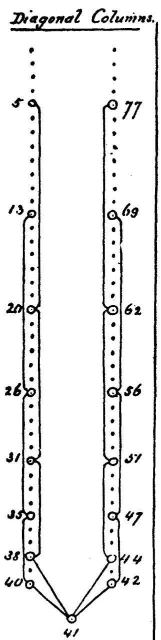

Fig. 111.

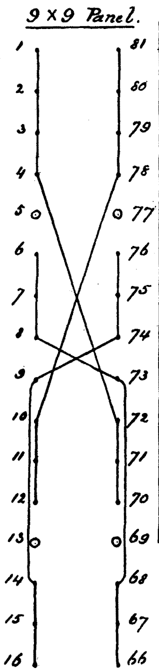

Fig. 112.

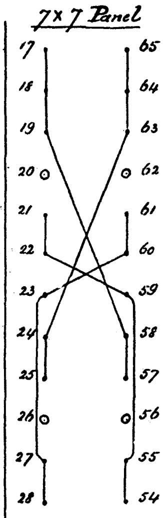

Fig. 113.

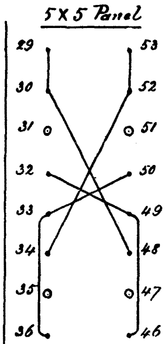

Fig. 114.

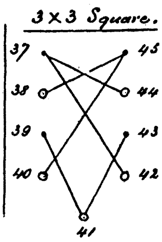

Fig. 115.

TOTALS:

3×3 square 123,
5×5 square 205,
7×7 square 287,
9×9 square 369.

77	1	2	3	4	72	71	70	69
76	62	17	18	19	58	57	56	6
75	61	51	29	30	48	47	21	7
74	60	50	44	37	42	32	22	8
9	23	33	39	41	43	49	59	73
14	27	36	40	45	38	46	55	68
15	28	35	53	52	34	31	54	67
16	26	65	64	63	24	25	20	66
13	81	80	79	78	10	11	12	5

Fig. 116.

that includes the series of numbers 1 to 49. The numbers for the two corner diagonal columns are indicated in their proper order by small circles in Fig. 106, and the arrangement of the panel numbers is given in Figs. 107, 108, and 109. As a final example of an

odd concentric square Fig. 116 shows one of 9×9, its development being given in Figs. 111, 112, 113, 114, and 115.

All these diagrams are simple and obvious expansions of those shown in Figs. 100 and 101 in connection with the 5×5 concentric square, and they and their numerous variations may be expanded

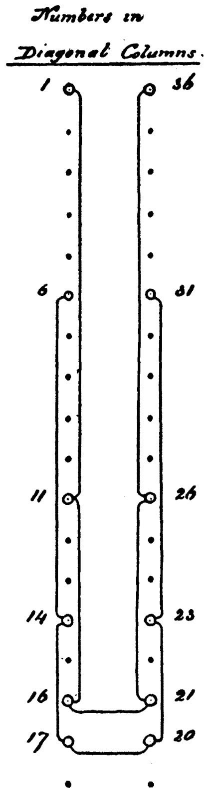

Fig. 117.

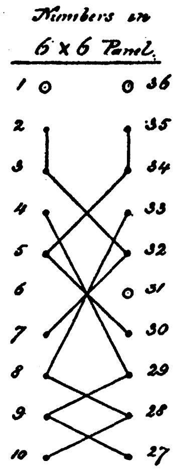

Fig. 118.

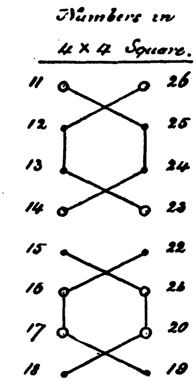

Fig. 119.

1	35	34	5	30	6
33	11	25	24	14	4
8	22	16	17	19	29
28	18	20	21	15	9
10	23	13	12	26	27
31	2	3	32	7	36

Fig. 120.

Totals of 4×4 square $= 74$.
Totals of 6×6 square $= 111$.

indefinitely and used for the construction of larger odd magic squares of this class.

The smallest even concentric magic square is that of 6×6, of which Fig. 120 is an example. The development of this square

may be traced in the diagrams given in Figs. 117, 118, and 119. The center square of 4×4 is perfect, but after the panel is added the enlarged square becomes imperfect as already noted. Figs. 121, 122, 123, and 124 illustrate another example of this square with diagrams of development.

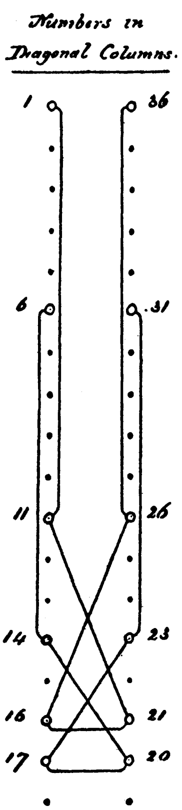

Fig. 121.

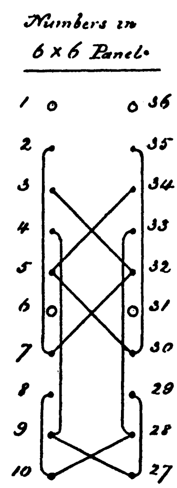

Fig. 122.

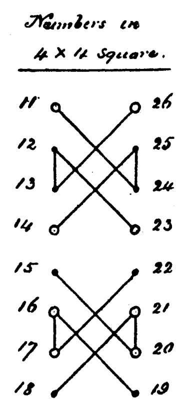

Fig. 123.

1	35	30	5	34	6
33	11	24	25	14	4
28	18	21	20	15	9
10	22	17	16	19	27
8	23	12	13	26	29
31	2	7	32	3	36

Fig. 124.

Totals of 4×4 square $= 74$.
Totals of 6×6 square $= 111$.

A concentric square of 8×8 with diagrams are given in Figs. 125, 126, 127, 128, and 129, and one of 10×10 in Figs. 130, 131, 132, 133, 134, and 135. It will be seen that all these larger squares have been developed in a very easy manner from successive expan-

sions of the diagrams used for the 6×6 square in Figs. 117, 118, and 119.

The rules governing the formation of concentric magic squares

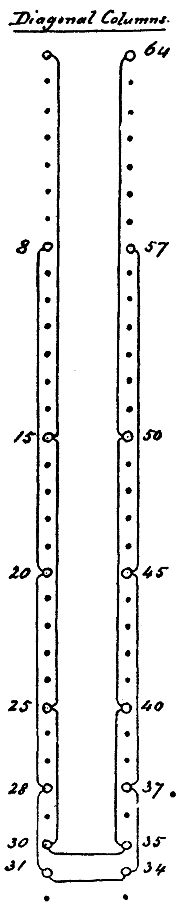

Fig. 125.

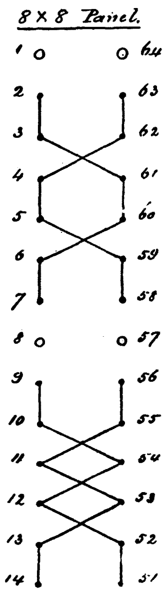

Fig. 126.

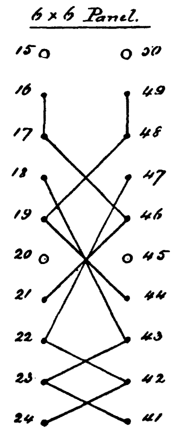

Fig. 127.

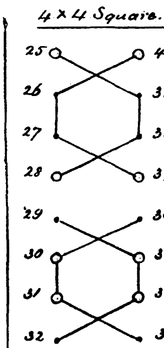

Fig. 128.

1	63	62	4	5	59	58	8
56	15	49	48	19	44	20	9
55	47	25	39	38	28	18	10
11	22	36	30	31	33	43	54
53	42	32	34	35	29	23	12
13	24	37	27	26	40	41	52
14	45	16	17	46	21	50	51
57	2	3	61	60	6	7	64

Fig. 129.

Totals of 4×4 square $= 130$.
Totals of 6×6 square $= 195$.
Totals of 8×8 square $= 260$.

have been hitherto considered somewhat difficult, but by the aid of diagrams as devised by the writer, their construction in great variety

and of any size has been reduced to an operation of extreme simplicity, involving only the necessary patience to construct the diagrams and copy the numbers.

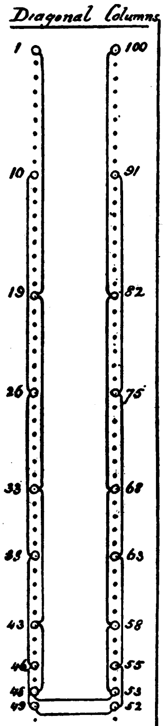

Fig. 130.

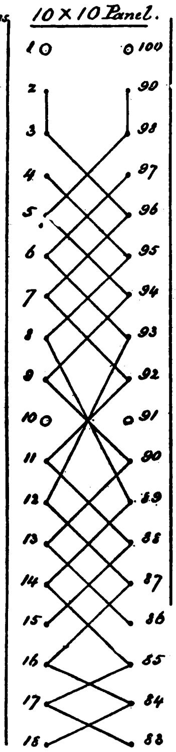

Fig. 131.

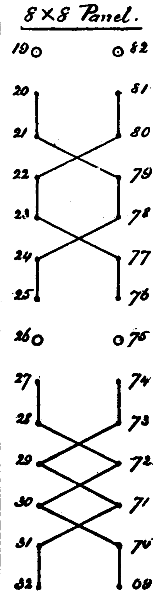

Fig. 132.

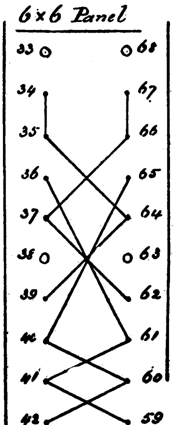

Fig. 133.

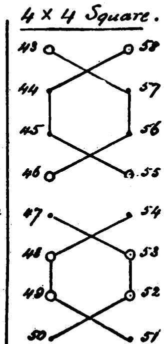

Fig. 134.

TOTALS:

4 × 4 square = 202
6 × 6 square = 303
8 × 8 square = 404
10 × 10 square = 505

1	99	98	5	94	9	90	13	86	10
97	19	81	80	22	23	77	76	26	4
6	74	33	67	66	37	62	38	27	95
93	73	65	43	57	56	46	36	28	8
12	29	40	54	48	49	51	61	72	89
87	71	60	50	52	53	47	41	30	14
16	31	42	55	45	44	58	59	70	85
84	32	63	34	35	64	39	68	69	17
18	75	20	21	79	78	24	25	82	83
91	2	3	96	7	92	11	88	15	100

Fig. 135.

GENERAL NOTES ON THE CONSTRUCTION OF MAGIC SQUARES.

There are two variables which govern the summations of all magic squares, viz.:

1. The Initial, or starting number.
2. The Increment, or increasing number.

When these two numbers are known, the summations can be easily determined, or when either of these variables and the summation are known, the other variable can be readily derived.

The most interesting problem in this connection is the construction of squares with predetermined summations, and this subject will therefore be first considered, assuming that the reader is familiar with the usual methods of building odd and even squares.

* * *

If a square of 3×3 is constructed in the usual manner, that is, beginning with unity and proceeding with regular increments of 1, the total of each column will be 15.

8	1	6
3	5	7
4	9	2

Totals = 15.

Fig. 136.

If 2 is used as the initial number instead of 1 and the square is again constructed with regular increments of 1, the total of each column will be 18 instead of 15.

9	2	7
4	6	8
5	10	3

Totals = 18.

Fig. 137.

If 2 is still used as the initial number and the square is once more constructed with regular increments of 2 instead of 1, the total of each column will be 30 instead of 18.

16	2	12
6	10	14
8	18	4

Totals $= 30$.

Fig. 138.

It therefore follows that there must be initial numbers, the use of which with given increments will entail summations of any predetermined amount, and there must also be increments, the use of which with given initial numbers, will likewise produce predetermined summations.

These initial numbers and increments may readily be determined by a simple form of equation which will establish a connection between them and the summation numbers.

Let:

$a =$ initial or starting number.

$b =$ increment.

$c =$ number of cells in one side of square.

$d =$ summation number when square is started with unity and built up with increments of 1.

$e =$ desired summation number.

Then:

$$(a \times c) + [(d - c) \times b] = e.$$

It will be found convenient to substitute a constant for $(d - c)$ in the foregoing equation and for this purpose a table of these constants is given below for all squares from 3×3 to 12×12.

Squares:	$(d - c) =$ Const. $=$ K
3×3	12
4×4	30
5×5	60
6×6	105
7×7	168
8×8	252
9×9	360
10×10	495
11×11	660
12×12	858

When using the above constants the equation will be:

$$(a \times c) + (\mathrm{K} \times b) = e.$$

EXAMPLES.

What initial number is required for the square of 3×3, with 1 as the increment, to produce 1903 as the summation?

Transposing the last equation:

$$\frac{e - (\mathrm{K} \times b)}{\mathrm{c}} = a,$$

or

$$\frac{1903 - (12 \times 1)}{3} = 630^1/_3 = \text{Initial No.}$$

637 1/3	630 1/3	635 1/3
632 1/3	634 1/3	636 1/3
633 1/3	638 1/3	631 1/3

Totals = 1903.

Fig. 139.

We will now apply the same example to a square of 4×4, in which case:

$$\frac{1903 - (30 \times 1)}{4} = 468\tfrac{1}{4} = \text{Initial No.}$$

468 1/4	482 1/4	481 1/4	471 1/4
479 1/4	473 1/4	474 1/4	476 1/4
475 1/4	477 1/4	478 1/4	472 1/4
480 1/4	470 1/4	469 1/4	483 1/4

Totals = 1903.

Fig. 140.

Also to a square of 5×5.

$$\frac{1903 - (60 \times 1)}{5} = 368.6 = \text{Initial No.}$$

384.6	391.6	368.6	375.6	382.6
390.6	372.6	374.6	381.6	383.6
371.6	373.6	380.6	387.6	389.6
377.6	379.6	386.6	388.6	370.6
378.6	385.6	392.6	369.6	376.6

Totals = 1903.

Fig. 141.

And for a square of 6×6.

$$\frac{1903 - (105 \times 1)}{6} = 299^2/_3 = \text{Initial No.}$$

$299\frac{2}{3}$	$333\frac{2}{3}$	$332\frac{2}{3}$	$301\frac{2}{3}$	$330\frac{2}{3}$	$304\frac{2}{3}$
$328\frac{2}{3}$	$306\frac{2}{3}$	$326\frac{2}{3}$	$325\frac{2}{3}$	$309\frac{2}{3}$	$305\frac{2}{3}$
$322\frac{2}{3}$	$321\frac{2}{3}$	$313\frac{2}{3}$	$314\frac{2}{3}$	$312\frac{2}{3}$	$317\frac{2}{3}$
$311\frac{2}{3}$	$315\frac{2}{3}$	$319\frac{2}{3}$	$320\frac{2}{3}$	$318\frac{2}{3}$	$316\frac{2}{3}$
$310\frac{2}{3}$	$324\frac{2}{3}$	$307\frac{2}{3}$	$308\frac{2}{3}$	$327\frac{2}{3}$	$323\frac{2}{3}$
$329\frac{2}{3}$	$300\frac{2}{3}$	$302\frac{2}{3}$	$331\frac{2}{3}$	$303\frac{2}{3}$	$334\frac{2}{3}$

Totals = 1903.

Fig. 142.

Squares built up with progressive increments of 1, have only thus far been considered. As before stated, this method can be varied by using increments greater or less than unity, but the same increment number must be used continuously throughout the construction of any given square.

EXAMPLES.

What initial number must be used in a square of 3×3, with increments of 3, to produce a summation of 1903?

Applying the equation given on page 578, but making $b = 3$ instead of 1, we have:

$$\frac{1903 - (12 \times 3)}{3} = 622^1/_3.$$

$622^1/_3$ is therefore the initial number and by using this in a 3×3 square with progressive increments of 3, the desired results are obtained.

$643\frac{1}{3}$	$622\frac{1}{3}$	$637\frac{1}{3}$
$628\frac{1}{3}$	$634\frac{1}{3}$	$640\frac{1}{3}$
$631\frac{1}{3}$	$646\frac{1}{3}$	$625\frac{1}{3}$

Totals = 1903.

Fig. 143.

To find the initial number with increments of 10.

$$\frac{1903 - (12 \times 10)}{3} = 594^1/_3 = \text{Initial No.}$$

664 1/3	594 1/3	644 1/3
614 1/3	634 1/3	654 1/3
624 1/3	674 1/3	604 1/3

Totals = 1903.

Fig. 144.

Or to find the initial number with increments of $^1/_3$.

$$\frac{1903-(12\times {}^1/_3)}{3}=633=\text{Initial No.}$$

635 1/3	633	634 2/3
633 2/3	634 1/3	635
634	635 2/3	633 1/3

Totals = 1903.

Fig. 145.

These examples being sufficient to illustrate the rule, we will pass on another step and show how to build squares with predetermined summations, using any desired initial numbers, with a proper increment.

EXAMPLES.

What increment number must be used in a square of 3×3, wherein 1 is the initial number and 1903 the desired summation?

Referring to equation on page 578 and transposing, we have:

$$\frac{e-(a\times c)}{K}=b=\text{Increment.}$$

or

$$\frac{1903-(1\times 3)}{12}=158{}^1/_3=\text{Increment.}$$

Starting therefore with unity and building up the square with successive increments of $158^1/_3$, we obtain the desired result.

1109 1/3	1	792 2/3
317 2/3	634 1/3	951
476	1267 2/3	159 1/3

Totals = 1903.

Fig. 146.

When it is desired to start with any number larger or smaller

than unity, the numbers in the equation can be modified accordingly. Thus if 4 is selected as an initial number, the equation will be:

$$\frac{1903-(4 \times 3)}{12} = 157^{7}/_{12} = \text{Increment.}$$

$1107\frac{1}{12}$	4	$791\frac{11}{12}$
$319\frac{2}{12}$	$634\frac{4}{12}$	$949\frac{6}{12}$
$476\frac{9}{12}$	$1264\frac{8}{12}$	$161\frac{7}{12}$

Totals = 1903.

Fig. 147.

or with an initial number of 5.

$$\frac{1903-(5 \times 3)}{12} = 157^{1}/_{8} = \text{Increment.}$$

$1106\frac{1}{3}$	5	$791\frac{2}{3}$
$319\frac{2}{3}$	$634\frac{1}{3}$	949
477	$1263\frac{2}{3}$	$162\frac{1}{3}$

Totals = 1903.

Fig. 148.

With an initial number of 500.

$$\frac{1903-(500 \times 3)}{12} = 33^{7}/_{12} = \text{Increment.}$$

$735\frac{1}{12}$	500	$677\frac{11}{12}$
$567\frac{2}{12}$	$634\frac{4}{12}$	$701\frac{6}{12}$
$600\frac{9}{12}$	$768\frac{8}{12}$	$533\frac{7}{12}$

Totals = 1903.

Fig. 149.

With an initial number of $^{1}/_{8}$.

$$\frac{1903-(^{1}/_{8} \times 3)}{12} = 158\tfrac{1}{2} = \text{Increment.}$$

$1109\frac{10}{12}$	$\frac{4}{12}$	$792\frac{10}{12}$
$317\frac{4}{12}$	$634\frac{4}{12}$	$951\frac{4}{12}$
$475\frac{10}{12}$	$1268\frac{4}{12}$	$158\frac{10}{12}$

Totals = 1903.

Fig. 150.

It is thus demonstrated that any initial number may be used

providing (in a square of 3 × 3) it is less than one-third of the summation. In a square of 4 × 4 it must be less than one-fourth of the summation, and so on.

To illustrate an extreme case, we will select 634 as an initial number in a 3 × 3 square and find the increment which will result in a summation of 1903.

$$\frac{1903 - (634 \times 3)}{12} = {}^1/_{12} \text{ Increment.}$$

$634\frac{7}{12}$	634	$634\frac{5}{12}$
$634\frac{2}{12}$	$634\frac{4}{12}$	$634\frac{6}{12}$
$634\frac{3}{12}$	$634\frac{8}{12}$	$634\frac{1}{12}$

Totals = 1903.

Fig. 151.

In the case of a square of 4 × 4, using 1 as a starting number and 1903 as a summation:

$$\frac{1903 - (1 \times 4)}{30} = 63.3 = \text{Increment.}$$

1	887.2	823.9	190.9
697.3	317.5	380.8	507.4
444.1	570.7	634	254.2
760.6	127.6	64.3	950.5

Totals = 1903.

Fig. 152.

As a final example of this rule we will select 475 as a starting number for a 4 × 4 square, the summation to be 1903.

$$\frac{1903 - (475 \times 4)}{30} = .1 = \text{Increment.}$$

475	476.4	476.3	475.3
476.1	475.5	475.6	475.8
475.7	475.9	476	475.4
476.2	475.2	475.1	476.5

Totals = 1903.

Fig. 153.

Having now considered the formation of magic squares with

predetermined summations by the use of proper initial numbers and increments. it only remains to show that the summation of any square may be found, when the initial number and the increment are given, by the application of the equation shown on page 578, viz.:

$$(a \times c) + (K \times b) = e.$$

EXAMPLES.

Find the summation number for a square of 3×3 using 5 as the initial number, and 7 as the increment.

$$(5 \times 3) + (12 \times 7) = 99 = \text{Summation.}$$

54	5	40
19	33	47
26	61	12

Totals = 99.

Fig. 154.

What will be the summation of a square of 4×4 using 9 as an initial number and 11 as an increment?

$$(9 \times 4) + (30 \times 11) = 366 = \text{Summation.}$$

9	163	152	42
130	64	75	97
86	108	119	53
141	31	20	174

Totals = 366.

Fig. 155 .

The preceding equations may also be used for the construction of magic squares involving zero and minus quantities, as illustrated in the following examples.

What will be the summation of a square of 3×3, using 10 as the initial number with —2 increments?

$$(10 \times 3) + (12 \times -2) = 6 = \text{Summation.}$$

-4	10	0
6	2	-2
4	-6	8

Totals = 6.

Fig. 156.

What initial number must be used in a square of 3×3 with increments of -3 to produce a summation of 3?

$$\frac{3 - (12 \times -3)}{3} = 13 = \text{Initial No.}$$

−8	13	−2
7	1	−5
4	−11	10

Totals = 3.

Fig. 157.

What initial number is required for a 3×3 square, with increments of 1, to produce a summation of 0?

$$\frac{0 - (12 \times 1)}{3} = -4 = \text{Initial No.}$$

3	−4	1
−2	0	2
−1	4	−3

Totals = 0.

Fig. 158.

What initial number is required for a 3×3 square, using increments of -4 to produce a summation of 0?

$$\frac{0 - (12 \times -4)}{3} = 16 = \text{Initial No.}$$

−12	16	−4
8	0	−8
4	−16	12

Totals = 0.

Fig. 159.

What initial number must be used in a square of 3×3 with increments of 1, to produce a summation of -6?

$$\frac{-6 - (12 \times 1)}{3} = -6 = \text{Initial No.}$$

1	−6	−1
−4	−2	0
−3	2	−5

Totals = −6.

Fig. 160.

What increment must be used in a square of 3×3 wherein -5 is the initial number, and 21 the required summation?

$$\frac{21 - (-5 \times 3)}{12} = 3 = \text{Increment.}$$

16	-5	10
1	7	13
4	19	-2

Totals = 21.

Fig. 161.

What increment must be used in a square of 3×3 wherein 12 is the initial number and -12 the required summation?

$$\frac{-12 - (12 \times 3)}{12} = -4 = \text{Increment.}$$

-16	12	-8
4	-4	-12
0	-20	8

Totals = -12.

Fig. 162.

What increment must be used in a square of 4×4 wherein 48 is the initial number and 42 the summation?

$$\frac{42 - (48 \times 4)}{30} = -5 = \text{Increment.}$$

48	-22	-17	33
-7	23	18	8
13	3	-2	28
-12	38	43	-27

Totals = 42.

Fig. 163.

The foregoing rules have been applied to examples in squares of small size only for the sake of brevity and simplicity, but the principles explained can evidently be expanded to any extent that may be desired.

Professor Scheffler and others have ingeniously applied some of the curious principles of the magic square to various figures such

as triangles, rectangles, pentagons, hexagons, etc., and magic cubes of various sizes have also been constructed.

It would be outside the scope of the present article to undertake the study of these interesting problems, but any who desire to learn something about them may find a brief description of same, with a few examples, in *Mathematical Essays and Recreations* by Hermann Schubert.*

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